
Nonlinear approximation with nonstationary Gabor frames

Emil Solsbæk Ottosen · Morten Nielsen

Abstract We consider sparseness properties of adaptive time-frequency representations obtained using nonstationary Gabor frames. Nonstationary Gabor frames generalize classical Gabor frames by allowing for adaptivity in either time or frequency. It is known that the concept of painless nonorthogonal expansions generalizes to the nonstationary case, providing perfect reconstruction and a FFT based implementation for compactly supported window functions sampled at a certain density. It is also known that for certain signal classes, nonstationary Gabor frames, with resolution varying in time, tend to provide sparser expansions than can be obtained with classical Gabor frames. In this article we show, for the continuous case, that sparseness of a nonstationary Gabor expansion is equivalent to smoothness in an associated decomposition space. In this way we characterize signals with sparse expansions relative to certain nonstationary Gabor frames. Based on this characterization we prove an upper bound on the approximation error occurring when thresholding the coefficients of the corresponding frame expansions. We complement the theoretical results with numerical experiments, estimating the rate of approximation obtained from thresholding the coefficients of both classical Gabor expansions and nonstationary Gabor expansions.

Keywords Time-frequency analysis, nonstationary Gabor frames, sparse frame expansions, decomposition spaces, nonlinear approximation

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1 Introduction

The field of Gabor theory [5, 19, 38] is concerned with representing signals as atomic decompositions using time-frequency localized atoms. The atoms are constructed as time-frequency shifts of a fixed window function according to some lattice parameters such that the resulting system constitutes a frame and, therefore, guarantees stable expansions [4, 30, 34]. Such frames are known under the name of Weyl-Heisenberg frames or Gabor frames and have been proven useful in a variety of applications [9, 22, 32]. The structure of Gabor frames implies a time-frequency resolution which depends only on the lattice parameters and the window function. In particular, the resolution is independent of the signal under consideration, which makes the corresponding implementation fast and easy to handle. The usage of a predetermined time-frequency resolution naturally raises the question of whether an improvement can be obtained by taking the signal class into consideration? This question has lead to many interesting approaches for constructing *adaptive* time-frequency representations [10, 11, 26, 39]. Unfortunately, for representations with resolution varying in *both* time and frequency there seems to be a trade-off between perfect reconstruction and fast implementation [29]. In this article, we therefore consider time-frequency representations with resolution varying in *either* time or frequency. The idea is to generalise the theory of painless nonorthogonal expansions [5] to the situation where multiple window functions are used along either the time- or the frequency axis. The resulting systems are known as (painless) *generalised shift-invariant systems* [24, 35] or (painless) *nonstationary Gabor frames* [1, 25].

We consider painless nonstationary Gabor frames (painless NSGF's) with resolution varying over time. In particular, we are interested in the corresponding implementation of so-called *scale frames* which have shown great potential in relation to analysis of music signals [1]. Given a signal, representing a piece of music, the idea is to calculate the onsets of the piece and then use short window functions around the onsets and long window functions between the onsets. This construction implies a time-frequency representation with good time resolution around the onsets (used to determine the tempo) and good frequency resolution between the onsets (used to determine the harmonics). As already noted in [1], scale frames tend to produce sparser representations than classical Gabor frames for certain classes of music signals. Sparseness of a time-frequency representation is desirable for several reasons [7, 18], mainly because it may reduce the computational cost for manipulating and storing the coefficients. Additionally, many signal classes are characterized by some kind of sparseness in time or frequency and the corresponding signals are, therefore, best described by a sparse time-frequency representation. For such signals, the task of feature identification also benefits from a sparse representation as the particular characteristics of the signal becomes easier to identify.

While modulation spaces [15, 16, 22] have turned out to be the proper function spaces for studying sparseness properties of classical Gabor frames, we need a more general framework for the nonstationary case. A NSGF (with resolution varying over time) corresponds to a sampling grid which is irregular over time but regular over frequency for each fixed time point. We therefore search for a smoothness space which is compatible with a (more or less) arbitrary partition of the time domain. Such a flexibility can be provided by decomposition spaces, as introduced by Feichtinger and Gröbner in [14, 17]. Decomposition spaces may be viewed as a generalization of the classical Wiener amalgam spaces [13, 23] but with no assumption of an upper bound on the measure of the members of the partition. Another way of stating this is that decomposition spaces are constructed using bounded *admissible* partitions of unity [17] instead of bounded *uniform* partitions of unity [13]. The partitions we consider are obtained by applying a set of invertible affine transformations $\{A_k(\cdot) + c_k\}_{k \in \mathbb{N}}$ on a fixed set $Q \subset \mathbb{R}^d$ [2].

In [31] we considered a similar approach, characterizing painless NSGF's, with resolution varying in frequency, using decomposition spaces. It is worth noting that there is a significant mathematical difference between decomposition spaces in time and in frequency. In this article, we introduce decomposition spaces in Section 2, and in Section 3 we show, based on the ideas in [31], how to construct a suitable decomposition space for a given painless NSGF with resolution varying in time. Then, in Section 4 we characterize the signals with sparse frame expansions and we prove an upper bound on the approximation rate occurring when thresholding the frame coefficients. Finally, in Section 5 we present the numerical results and in Section 6 we give the conclusions.

Let us now briefly go through our notation. We let $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$ denote the Fourier transform with the usual extension to $L^2(\mathbb{R}^d)$. By $F \asymp G$ we mean that there exist two constants $0 < C_1, C_2 < \infty$ such that $C_1 F \leq G \leq C_2 F$. For two normed vector spaces X and Y , $X \hookrightarrow Y$ means that $X \subset Y$ and $\|f\|_Y \leq C \|f\|_X$ for some constant C and all $f \in X$. We say that a non-empty open set $\Omega' \subset \mathbb{R}^d$ is *compactly contained* in an open set $\Omega \subset \mathbb{R}^d$ if $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact. We

call $\{x_i\}_{i \in \mathcal{I}} \subset \mathbb{R}^d$ a δ -separated set if $\inf_{j,k \in \mathcal{I}, j \neq k} \|x_j - x_k\|_2 = \delta > 0$. Finally, by I_d we denote the identity operator on \mathbb{R}^d and by χ_Q we denote the indicator function for a set $Q \subset \mathbb{R}^d$.

2 Decomposition spaces

In this section we define decomposition spaces [17], based on structured coverings [2], and we prove several important properties of such spaces. For an invertible matrix $A \in GL(\mathbb{R}^d)$, and a constant $c \in \mathbb{R}^d$, we define the affine transformation $Tx = Ax + c$ with $x \in \mathbb{R}^d$. Given a family $\mathcal{T} = \{A_k(\cdot) + c_k\}_{k \in \mathbb{N}}$ of invertible affine transformations on \mathbb{R}^d , and a subset $Q \subset \mathbb{R}^d$, we let $\{Q_T\}_{T \in \mathcal{T}} := \{T(Q)\}_{T \in \mathcal{T}}$ and

$$\tilde{\mathcal{T}} := \{T' \in \mathcal{T} \mid Q_{T'} \cap Q_T \neq \emptyset\}, \quad T \in \mathcal{T}. \quad (1)$$

We say that $\mathcal{Q} := \{Q_T\}_{T \in \mathcal{T}}$ is an *admissible covering* of \mathbb{R}^d if $\bigcup_{T \in \mathcal{T}} Q_T = \mathbb{R}^d$ and there exists $n_0 \in \mathbb{N}$ such that $|\tilde{\mathcal{T}}| \leq n_0$ for all $T \in \mathcal{T}$.

Definition 1 (\mathcal{Q} -moderate weight). Let $\mathcal{Q} := \{Q_T\}_{T \in \mathcal{T}}$ be an admissible covering. A function $u : \mathbb{R}^d \rightarrow (0, \infty)$ is called \mathcal{Q} -moderate if there exists $C > 0$ such that $u(x) \leq Cu(y)$ for all $x, y \in Q_T$ and all $T \in \mathcal{T}$. A \mathcal{Q} -moderate weight (derived from u) is a sequence $\{\omega_T\}_{T \in \mathcal{T}} := \{u(x_T)\}_{T \in \mathcal{T}}$ with $x_T \in Q_T$ for all $T \in \mathcal{T}$.

For the rest of this article, we shall use the explicit choice $u(x) := 1 + \|x\|$ for the function u in Definition 1. Let us now define structured coverings [2] of the time domain.

Definition 2 (Structured covering). Given a family $\mathcal{T} = \{A_k(\cdot) + c_k\}_{k \in \mathbb{N}}$ of invertible affine transformations on \mathbb{R}^d , suppose there exist two bounded open sets $P \subset Q \subset \mathbb{R}^d$, with P compactly contained in Q , such that

1. $\{P_T\}_{T \in \mathcal{T}}$ and $\{Q_T\}_{T \in \mathcal{T}}$ are admissible coverings.
2. There exists a δ -separated set $\{x_T\}_{T \in \mathcal{T}} \subset \mathbb{R}^d$, with $x_T \in Q_T$ for all $T \in \mathcal{T}$, such that $\{\omega_T\}_{T \in \mathcal{T}} := \{u(x_T)\}_{T \in \mathcal{T}}$ is a \mathcal{Q} -moderate weight.

Then we call $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ a *structured covering*.

For a structured covering we have the associated concept of a *bounded admissible partition of unity* (BAPU) [17].

Definition 3 (BAPU). Let $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ be a structured covering of \mathbb{R}^d . A corresponding BAPU subordinate to \mathcal{Q} is a family of non-negative functions $\{\psi_T\}_{T \in \mathcal{T}} \subset C_c^\infty(\mathbb{R}^d)$ satisfying

1. $\text{supp}(\psi_T) \subset Q_T$, $\forall T \in \mathcal{T}$.
2. $\sum_{T \in \mathcal{T}} \psi_T(x) = 1$, $\forall x \in \mathbb{R}^d$.
3. $\sup_{T \in \mathcal{T}} \|\psi_T\|_{L^\infty} < \infty$.

Given a structured covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$, we can always construct a subordinate BAPU. Choose a non-negative function $\Phi \in C_c^\infty(\mathbb{R}^d)$, with $\Phi(x) = 1$ for all $x \in P$ and $\text{supp}(\Phi) \subset Q$, and define

$$\psi_T(x) := \frac{\Phi(T^{-1}x)}{\sum_{T' \in \mathcal{T}} \Phi(T'^{-1}x)}, \quad x \in \mathbb{R}^d,$$

for all $T \in \mathcal{T}$. With this construction, it is clear that Definition 3(1) is satisfied. Further, since $\{P_T\}_{T \in \mathcal{T}}$ is an admissible covering, $1 \leq \sum_{T' \in \mathcal{T}} \Phi(T'^{-1}x) \leq n_0$ for all $x \in \mathbb{R}^d$, which shows that Definition 3(2) and Definition 3(3) hold.

Remark 1 We note that the assumption in Definition 2(2) is not necessary for constructing a subordinate BAPU, however, the assumption is needed for proving Theorem 1.

Given a structured covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ with \mathcal{Q} -moderate weight $\{\omega_T\}_{T \in \mathcal{T}}$ and BAPU $\{\psi_T\}_{T \in \mathcal{T}}$. For $s \in \mathbb{R}$ and $1 \leq q \leq \infty$, we define the associated weighted sequence space

$$\ell_{\omega^s}^q(\mathcal{T}) := \left\{ \{a_T\}_{T \in \mathcal{T}} \subset \mathbb{C} \mid \|\{a_T\}_{T \in \mathcal{T}}\|_{\ell_{\omega^s}^q} := \|\{\omega_T^s a_T\}_{T \in \mathcal{T}}\|_{\ell^q} < \infty \right\}.$$

Given $\{a_T\}_{T \in \mathcal{T}} \in \ell_{\omega^s}^q(\mathcal{T})$, we define $\{a_T^+\}_{T \in \mathcal{T}}$ by $a_T^+ := \sum_{T' \in \tilde{\mathcal{T}}} a_{T'}$. Since $\{\omega_T\}_{T \in \mathcal{T}}$ is \mathcal{Q} -moderate, $\{a_T\}_{T \in \mathcal{T}} \rightarrow \{a_T^+\}_{T \in \mathcal{T}}$ defines a bounded operator on $\ell_{\omega^s}^q(\mathcal{T})$ according to [17, Remark 2.13 and Lemma 3.2]. Denoting its operator norm by C_+ , we have

$$\|\{a_T^+\}_{T \in \mathcal{T}}\|_{\ell_{\omega^s}^q} \leq C_+ \|\{a_T\}_{T \in \mathcal{T}}\|_{\ell_{\omega^s}^q}, \quad \forall \{a_T\}_{T \in \mathcal{T}} \in \ell_{\omega^s}^q(\mathcal{T}). \quad (2)$$

We now define decomposition spaces as first introduced in [17].

Definition 4 (Decomposition space). Let $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ be a structured covering with \mathcal{Q} -moderate weight $\{\omega_T\}_{T \in \mathcal{T}}$ and BAPU $\{\psi_T\}_{T \in \mathcal{T}}$. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define the *decomposition space* $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ as the set of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\|f\|_{D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)} := \|\{\|\psi_T f\|_{L^p}\}_{T \in \mathcal{T}}\|_{\ell_{\omega^s}^q} < \infty.$$

Remark 2 According to [17, Theorem 3.7], $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ is independent of the particular choice of BAPU and different choices yield equivalent norms. Actually the results in [17] show that $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ is invariant under certain geometric modifications of \mathcal{Q} , but we will not go into detail here.

Remark 3 In contrast to the approach taken in [31] (where the decomposition is performed on the frequency side), we do not allow $p, q < 1$ in Definition 4 since a simple consideration shows that the resulting decomposition spaces would not be complete in this case.

We now consider some familiar examples of decomposition spaces. By standard arguments it is easy to verify that $D(\mathcal{Q}, L^2, \ell^2) = L^2(\mathbb{R}^d)$ with equivalent norms for any structured covering \mathcal{Q} . The next example shows how to construct Wiener amalgam spaces.

Example 1 Let $Q \in \mathbb{R}^d$ be an open cube with center 0 and side-length $r > 1$. Define $\mathcal{T} := \{T_k\}_{k \in \mathbb{Z}^d}$, with $T_k x := x - k$ for all $k \in \mathbb{Z}^d$, and let $\{k\}_{k \in \mathbb{Z}^d}$ be the δ -separated set from Definition 2(2). With $\mathcal{Q} := \{Q_T\}_{T \in \mathcal{T}}$, $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ corresponds to $W(L^p, \ell_{\omega^s}^q)$ for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, see [13] for further details. \square

Let us now prove the following important properties of decomposition spaces.

Theorem 1 Given a structured covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ with \mathcal{Q} -moderate weight $\{\omega_T\}_{T \in \mathcal{T}}$ and subordinate BAPU $\{\psi_T\}_{T \in \mathcal{T}}$. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$,

1. $\mathcal{S}(\mathbb{R}^d) \hookrightarrow D(\mathcal{Q}, L^p, \ell_{\omega^s}^q) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.
2. $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ is a Banach space.
3. If $p, q < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$.
4. If $p, q < \infty$, then the dual space of $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ can be identified with $D(\mathcal{Q}, L^{p'}, \ell_{\omega^{-s}}^{q'})$ with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

The proof of Theorem 1 can be found in Appendix A. In the next section we construct decomposition spaces, which are compatible with the structure of certain nonstationary Gabor frames.

3 Nonstationary Gabor frames

In this section, we construct nonstationary Gabor frames with flexible time resolution using the notation of [1]. Given a set of window functions $\{g_n\}_{n \in \mathbb{Z}^d}$ in $L^2(\mathbb{R}^d)$, with corresponding frequency sampling steps $b_n > 0$, then for $m, n \in \mathbb{Z}^d$ we define atoms of the form

$$g_{m,n}(x) := g_n(x) e^{2\pi i m b_n \cdot x}, \quad x \in \mathbb{R}^d. \quad (3)$$

The choice of \mathbb{Z}^d as index set for n is only a matter of notational convenience; any countable index set would do.

Example 2 With $g_n(x) := g(x - na)$ and $b_n := b$ for all $n \in \mathbb{Z}^d$ we get

$$g_{m,n}(x) := g(x - na) e^{2\pi i m b \cdot x}, \quad x \in \mathbb{R}^d, \quad (4)$$

which just corresponds to a standard Gabor system. \square

If $\sum_{m,n} |\langle f, g_{m,n} \rangle|^2 \asymp \|f\|_2^2$ for all $f \in L^2(\mathbb{R}^d)$, we refer to $\{g_{m,n}\}_{m,n}$ as a *nonstationary Gabor frame* (NSGF). For an NSGF $\{g_{m,n}\}_{m,n}$, the frame operator

$$Sf = \sum_{m,n \in \mathbb{Z}^d} \langle f, g_{m,n} \rangle g_{m,n}, \quad f \in L^2(\mathbb{R}^d),$$

is invertible and we have the expansions

$$f = \sum_{m,n \in \mathbb{Z}^d} \langle f, g_{m,n} \rangle \tilde{g}_{m,n}, \quad f \in L^2(\mathbb{R}^d),$$

with $\{\tilde{g}_{m,n}\}_{m,n} := \{S^{-1} g_{m,n}\}_{m,n}$ being the canonical dual frame of $\{g_{m,n}\}_{m,n}$. According to [1, Theorem 1], we have the following result.

Theorem 2 Let $\{g_n\}_{n \in \mathbb{Z}^d} \subset L^2(\mathbb{R}^d)$ with frequency sampling steps $\{b_n\}_{n \in \mathbb{Z}^d}$, $b_n > 0$ for all $n \in \mathbb{Z}^d$. Assuming $\text{supp}(g_n) \subseteq [0, \frac{1}{b_n}]^d + a_n$, with $a_n \in \mathbb{R}^d$ for all $n \in \mathbb{Z}^d$, the frame operator for the system

$$g_{m,n}(x) = g_n(x) e^{2\pi i m b_n \cdot x}, \quad \forall m, n \in \mathbb{Z}^d, \quad x \in \mathbb{R}^d,$$

is given by

$$Sf(x) = \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{b_n^d} |g_n(x)|^2 \right) f(x), \quad f \in L^2(\mathbb{R}^d).$$

The system $\{g_{m,n}\}_{m,n \in \mathbb{Z}^d}$ constitutes a frame for $L^2(\mathbb{R}^d)$, with frame-bounds $0 < A \leq B < \infty$, if and only if

$$A \leq \sum_{n \in \mathbb{Z}^d} \frac{1}{b_n^d} |g_n(x)|^2 \leq B, \quad \text{for a.e. } x \in \mathbb{R}^d, \quad (5)$$

and the canonical dual frame is then given by

$$\tilde{g}_{m,n}(x) = \frac{g_n(x)}{\sum_{l \in \mathbb{Z}^d} \frac{1}{b_l^d} |g_l(x)|^2} e^{2\pi i m b_n \cdot x}, \quad x \in \mathbb{R}^d. \quad (6)$$

Remark 4 We note that the canonical dual frame in (6) possesses the same structure as the original frame, which is a property not shared by general NSGF's. We also note that the canonical tight frame can be obtained by taking the square root of the denominator in (6).

Traditionally, an NSGF satisfying the assumptions of Theorem 2 is called a *painless* NSGF, referring to the fact that the frame operator is a simple multiplication operator. This terminology is adopted from the classical *painless nonorthogonal expansions* [5], which corresponds to the painless case for classical Gabor frames. By slight abuse of notation we use the term "painless" to denote the NSGF's satisfying Definition 5 below. In order to properly formulate this definition, we first need some preliminary notation which we adopt from [31].

Let $\{g_n\}_{n \in \mathbb{Z}^d} \subset L^2(\mathbb{R}^d)$ satisfy the assumptions in Theorem 2. Given $C_* > 0$ we denote by $\{I_n\}_{n \in \mathbb{Z}^d}$ the open cubes

$$I_n := \left(-\varepsilon_n, \frac{1}{b_n} + \varepsilon_n \right)^d + a_n, \quad \forall n \in \mathbb{Z}^d, \quad (7)$$

with $\varepsilon_n := C_*/b_n$ for all $n \in \mathbb{Z}^d$. We note that $\text{supp}(g_{m,n}) \subset I_n$ for all $m, n \in \mathbb{Z}^d$. For $n \in \mathbb{Z}^d$ we define

$$\tilde{n} := \left\{ n' \in \mathbb{Z}^d \mid I_{n'} \cap I_n \neq \emptyset \right\},$$

using the notation of (1). We recall that $u(x) := 1 + \|x\|$.

Definition 5 (Painless NSGF). Let $\{g_n\}_{n \in \mathbb{Z}^d} \subset L^2(\mathbb{R}^d)$ satisfy the assumptions in Theorem 2, and assume that,

1. There exists $C_* > 0$ and $n_0 \in \mathbb{N}$, such that the open cubes $\{I_n\}_{n \in \mathbb{Z}^d}$ satisfy $|\tilde{n}| \leq n_0$ uniformly for all $n \in \mathbb{Z}^d$.
2. $\{a_n\}_{n \in \mathbb{Z}^d}$ forms a δ -separated set and $\{u(a_n)\}_{n \in \mathbb{Z}^d}$ constitutes a $\{I_n\}_{n \in \mathbb{Z}^d}$ -moderate weight.
3. The g_n 's are continuous, real valued and satisfy

$$g_n(x) \leq C b_n^{d/2} \chi_{I_n}(x), \quad \text{for all } n \in \mathbb{Z}^d,$$

for some uniform constant $C > 0$.

Then we refer to $\{g_{m,n}\}_{m,n \in \mathbb{Z}^d}$ as a *painless* NSGF.

The assumptions in Definition 5 are easily satisfied, but the support conditions in Theorem 2 are rather restrictive and implies a certain redundancy of the system. Nevertheless, we must assume some structure on the dual frame, which is not provided by general NSGF's. We choose the framework of painless NSGF's and base our arguments on the fact that the dual frame possess the same structure as the original frame. We expect it is possible to extend the theory developed in this article to a more general settings by imposing general existence results for NSGF's [12, 25, 37]. We now provide a simple example of a set of window functions satisfying Definition 5(3).

Example 3 Choose a continuous real valued function $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$ with $\text{supp}(\varphi) \subseteq [0, 1]^d$. For $n \in \mathbb{Z}^d$ define

$$g_n(x) := b_n^{d/2} \varphi(b_n(x - a_n)), \quad x \in \mathbb{R}^d,$$

with $a_n \in \mathbb{R}^d$ and $b_n > 0$. Then $\text{supp}(g_n) \subseteq [0, \frac{1}{b_n}] + a_n$ and Definition 5(3) is satisfied. \square

Following the approach taken in [31], we define $\mathcal{Q} := (0, 1)^d$ together with the set of affine transformations $\mathcal{T} := \{A_n(\cdot) + c_n\}_{n \in \mathbb{Z}^d}$ with

$$A_n := \left(2\varepsilon_n + \frac{1}{b_n} \right) \cdot I_d, \quad \text{and} \quad (c_n)_j = -\varepsilon_n + (a_n)_j, \quad 1 \leq j \leq d.$$

It is then easily shown that $\mathcal{Q} := \{Q_T\}_{T \in \mathcal{T}} = \{I_n\}_{n \in \mathbb{Z}^d}$ forms a structured covering of \mathbb{R}^d [31, Lemma 4.1]. Given $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we may therefore construct the associated decomposition space $D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$. For notational convenience we change notation and write $\{g_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}}$ such that $\text{supp}(g_{m,T}) \subset Q_T$ for all $m \in \mathbb{Z}^d$ and all $T \in \mathcal{T}$.

4 Characterization of decomposition spaces

Using the notation of [2] we define the sequence space $d(\mathcal{Q}, \ell^p, \ell_{\omega^s}^q)$ as the set of coefficients $\{c_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \subset \mathbb{C}$ satisfying

$$\left\| \{c_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^p, \ell_{\omega^s}^q)} := \left\| \left\{ \left\| \{c_{m,T}\}_{m \in \mathbb{Z}^d} \right\|_{\ell^p} \right\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^s}^q} < \infty,$$

for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. We can now prove the following important stability result.

Theorem 3 *Let $s \in \mathbb{R}$ and $1 \leq p \leq 2$. For $f \in D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ and $1 \leq q \leq \infty$,*

$$\left\| \{\langle f, g_{m,T} \rangle\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^{p'}, \ell_{\omega^s}^q)} \leq C \|f\|_{D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)}, \quad (8)$$

and for $h \in D(\mathcal{Q}, L^{p'}, \ell_{\omega^s}^q)$ and $1 \leq q < \infty$,

$$\|h\|_{D(\mathcal{Q}, L^{p'}, \ell_{\omega^s}^q)} \leq C' \left\| \{\langle h, g_{m,T} \rangle\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^p, \ell_{\omega^s}^q)}, \quad (9)$$

with $1/p + 1/p' = 1$.

Proof We first prove (8). Given $f \in D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$, since $\widetilde{\psi}_T := \sum_{T' \in \tilde{\mathcal{T}}} \psi_{T'} \equiv 1$ on \mathcal{Q}_T , then

$$\begin{aligned} \left\| \{\langle f, g_{m,T} \rangle\}_{m \in \mathbb{Z}^d} \right\|_{\ell^{p'}} &= \left(\sum_{m \in \mathbb{Z}^d} |\langle \widetilde{\psi}_T f, g_{m,T} \rangle|^{p'} \right)^{1/p'} \\ &= b_T^{-d/2} \left(\sum_{m \in \mathbb{Z}^d} \left| b_T^{d/2} \int_{\mathbb{R}^d} \widetilde{\psi}_T(x) f(x) \overline{g_T(x)} e^{-2\pi i m b_T \cdot x} dx \right|^{p'} \right)^{1/p'}, \end{aligned}$$

with $b_T > 0$ being the frequency sampling step. Since $1 \leq p \leq 2$ we can use the Hausdorff-Young inequality [27, Theorem 2.1 on page 98], which together with Definition 5(3) imply

$$\left\| \{\langle f, g_{m,T} \rangle\}_{m \in \mathbb{Z}^d} \right\|_{\ell^{p'}} \leq b_T^{-d/2} \|\widetilde{\psi}_T f g_T\|_{L^p} \leq C_1 \|\widetilde{\psi}_T f\|_{L^p}.$$

Hence, using (2) we get

$$\left\| \{\langle f, g_{m,T} \rangle\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^{p'}, \ell_{\omega^s}^q)} \leq C_1 \left\| \{\|\widetilde{\psi}_T f\|_{L^p}\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^s}^q} \leq C_2 \|f\|_{D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)}.$$

Let us now prove (9). Given $h \in D(\mathcal{Q}, L^{p'}, \ell_{\omega^s}^q)$ and $\sigma \in \mathcal{S}(\mathbb{R}^d)$, we write the frame expansion of σ with respect to $\{g_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}}$ and apply Hölder's inequality twice to obtain

$$\begin{aligned} |\langle h, \sigma \rangle| &\leq \sum_{T \in \mathcal{T}} \sum_{m \in \mathbb{Z}^d} |\langle \sigma, g_{m,T} \rangle \langle h, \tilde{g}_{m,T} \rangle| \\ &\leq \sum_{T \in \mathcal{T}} \left(\sum_{m \in \mathbb{Z}^d} |\langle \sigma, g_{m,T} \rangle|^{p'} \omega_T^{-p's} \right)^{1/p'} \left(\sum_{m \in \mathbb{Z}^d} |\langle h, \tilde{g}_{m,T} \rangle|^p \omega_T^{ps} \right)^{1/p} \\ &\leq \left\| \{\langle \sigma, g_{m,T} \rangle\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^{p'}, \ell_{\omega^{-s}}^{q'})} \left\| \{\langle h, \tilde{g}_{m,T} \rangle\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^p, \ell_{\omega^s}^q)}. \end{aligned} \quad (10)$$

According to (8) then

$$\left\| \{\langle \sigma, g_{m,T} \rangle\}_{m,T} \right\|_{d(\mathcal{Q}, \ell^{p'}, \ell_{\omega^{-s}}^{q'})} \leq C \|\sigma\|_{D(\mathcal{Q}, L^p, \ell_{\omega^{-s}}^{q'})} < \infty,$$

since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow D(\mathcal{Q}, L^p, \ell_{\omega^{-s}}^{q'})$. Finally, since the dual space of $D(\mathcal{Q}, L^{p'}, \ell_{\omega^s}^q)$ can be identified with $D(\mathcal{Q}, L^p, \ell_{\omega^{-s}}^{q'})$, and since $\mathcal{S}(\mathbb{R}^d)$ is dense in $D(\mathcal{Q}, L^p, \ell_{\omega^{-s}}^{q'})$, (10) implies

$$\begin{aligned} \|h\|_{D(\mathcal{Q}, L^{p'}, \ell_{\omega^s}^q)} &= \sup_{\|\sigma\|_{D(\mathcal{Q}, L^p, \ell_{\omega^{-s}}^{q'})}=1} |\langle h, \sigma \rangle| \leq C_1 \left\| \{ \langle h, \tilde{g}_{m,T} \rangle \}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^p, \ell_{\omega^s}^q)} \\ &\leq C_2 \left\| \{ \langle h, g_{m,T} \rangle \}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^p, \ell_{\omega^s}^q)}, \end{aligned}$$

where we use (5) in the last inequality. This completes the proof of (9). \square

We note that for $p = 2$, Theorem 3 yields the equivalence

$$\|f\|_{D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)} \asymp \left\| \{ \langle f, g_{m,T} \rangle \}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \right\|_{d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q)}, \quad f \in D(\mathcal{Q}, L^2, \ell_{\omega^s}^q),$$

for $s \in \mathbb{R}$ and $1 \leq q < \infty$. It follows that the *coefficient operator* $C: f \rightarrow \{ \langle f, g_{m,T} \rangle \}_{m,T}$ is bounded from $D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)$ into $d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q)$. We define the corresponding *reconstruction operator* as

$$R\left(\{c_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}}\right) = \sum_{T \in \mathcal{T}} \sum_{m \in \mathbb{Z}^d} c_{m,T} \tilde{g}_{m,T}, \quad \forall \{c_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \in d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q).$$

With this notation we have the following result.

Proposition 1 *Given $s \in \mathbb{R}$ and $1 \leq q < \infty$, the reconstruction operator R is bounded from $d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q)$ onto $D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)$ and we have the expansions*

$$f = RC(f) = \sum_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \langle f, g_{m,T} \rangle \tilde{g}_{m,T}, \quad f \in D(\mathcal{Q}, L^2, \ell_{\omega^s}^q), \quad (11)$$

with unconditional convergence.

Proof We first prove that R is bounded. Given $\{c_{m,T}\}_{m,T} \in d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q)$, (2) yields

$$\begin{aligned} \|R(\{c_{m,T}\}_{m,T})\|_{D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)} &= \left\| \left\{ \left\| \psi_T \left(\sum_{T' \in \mathcal{T}} \sum_{m \in \mathbb{Z}^d} c_{m,T'} \tilde{g}_{m,T'} \right) \right\|_{L^2} \right\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^s}^q} \\ &\leq C_1 \left\| \left\{ \left\| \sum_{m \in \mathbb{Z}^d} c_{m,T} g_{m,T} \right\|_{L^2} \right\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^s}^q}. \end{aligned} \quad (12)$$

Applying Definition 5(3) and the Hausdorff-Young inequality [27, Theorem 2.2 on page 99] we get

$$\left\| \sum_{m \in \mathbb{Z}^d} c_{m,T} g_{m,T} \right\|_{L^2}^2 \leq C \int_{\mathbb{R}^d} \left| b_T^{d/2} \sum_{m \in \mathbb{Z}^d} c_{m,T} e^{2\pi i m b_T \cdot x} \right|^2 dx \leq C \|\{c_{m,T}\}_{m \in \mathbb{Z}^d}\|_{\ell^2}^2. \quad (13)$$

Combining (12) and (13) we arrive at

$$\begin{aligned} \|R(\{c_{m,T}\}_{m,T})\|_{D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)} &\leq C_2 \left\| \left\{ \|\{c_{m,T}\}_{m \in \mathbb{Z}^d}\|_{\ell^2} \right\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^s}^q} \\ &= C_2 \|\{c_{m,T}\}_{m,T}\|_{d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q)}, \end{aligned} \quad (14)$$

which shows the boundedness of R . Let us now prove the unconditional convergence of (11). Given $f \in D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)$ we can find a sequence $\{f_k\}_{k \geq 1} \in \mathcal{S}(\mathbb{R}^d)$ such that $f_k \rightarrow f$ in $D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)$. For each k we have the expansion $f_k = RC(f_k)$, and by continuity of RC we get $f = RC(f)$. Given $\varepsilon > 0$, (14) implies that we can find a *finite* subset $F_0 \subseteq \mathbb{Z}^d \times \mathcal{T}$, such that for all finite sets $F \supseteq F_0$,

$$\left\| f - \sum_{(m,T) \in F} \langle f, g_{m,T} \rangle \tilde{g}_{m,T} \right\|_{D(\mathcal{Q}, L^2, \ell_{\omega^s}^q)} \leq C \left\| \{ \langle f, g_{m,T} \rangle \}_{(m,T) \notin F} \right\|_{d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^q)} < \varepsilon.$$

According to [22, Proposition 5.3.1 on page 98], this property is equivalent to unconditional convergence. \square

Based on Proposition 1, we can show some important properties of $\{g_{m,T}\}_{m \in \mathbb{Z}^d, T \in \mathcal{T}}$ in connection with nonlinear approximation theory [6]. Assume $f \in D(\mathcal{Q}, L^2, \ell_{\omega^s}^2)$, for $s \in \mathbb{R}$, and write the frame expansion

$$f = \sum_{m \in \mathbb{Z}^d, T \in \mathcal{T}} \langle f, g_{m,T} \rangle \tilde{g}_{m,T}. \quad (15)$$

Let $\{\theta_k\}_{k \in \mathbb{N}}$ be a rearrangement of the frame coefficients $\{\langle f, g_{m,T} \rangle\}_{m,T}$ such that $\{|\theta_k|\}_{k \in \mathbb{N}}$ constitutes a decreasing sequence. Also, let f_N be the N -term approximation to f obtained by extracting the terms in (15) corresponding to the N largest coefficients $\{\theta_k\}_{k=1}^N$. Since R is bounded, [20, Theorem 6] implies that for each $1 \leq \tau < 2$,

$$\|f - f_N\|_{D(\mathcal{Q}, L^2, \ell_{\omega^s}^2)} \leq C_1 \left\| \{\theta_k\}_{k > N} \right\|_{d(\mathcal{Q}, \ell^2, \ell_{\omega^s}^2)} \leq C_2 N^{-\alpha} \left\| \{\theta_k\}_{k \in \mathbb{N}} \right\|_{d(\mathcal{Q}, \ell^\tau, \ell_{\omega^s}^\tau)}, \quad (16)$$

with $\alpha := 1/\tau - 1/2$. According to (8), the coefficient operator is bounded from $D(\mathcal{Q}, L^{\tau'}, \ell_{\omega^s}^\tau)$ into $d(\mathcal{Q}, \ell^\tau, \ell_{\omega^s}^\tau)$, so we may continue on (16) and write

$$\|f - f_N\|_{D(\mathcal{Q}, L^2, \ell_{\omega^s}^2)} \leq C_3 N^{-\alpha} \|f\|_{D(\mathcal{Q}, L^{\tau'}, \ell_{\omega^s}^\tau)}, \quad (17)$$

with $1/\tau + 1/\tau' = 1$. We conclude that for $f \in D(\mathcal{Q}, L^2, \ell_{\omega^s}^2) \cap D(\mathcal{Q}, L^{\tau'}, \ell_{\omega^s}^\tau)$, we obtain good approximations in $D(\mathcal{Q}, L^2, \ell_{\omega^s}^2)$ by thresholding the frame coefficients in (15). The rate of the approximation is given by $\alpha \in (0, 1/2]$.

5 Numerical experiments

In this section we provide the numerical experiments, thresholding coefficients of both stationary and nonstationary Gabor expansions. We consider a signal of length 262144 consisting of a short piece of music recorded on a grand piano and sampled at 44100 Hz¹. The music is the beginning of "Georgia On My Mind" by Ray Charles and the corresponding music sheet can be found in Figure 1.

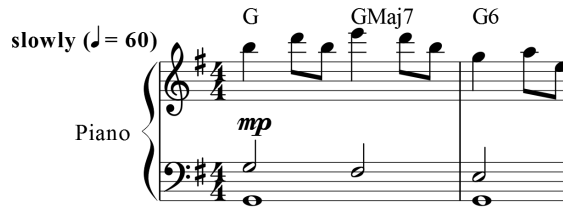


Fig. 1 Music sheet for "Georgia On My Mind" by Ray Charles.

¹ The corresponding sound file can be found at <http://homepage.univie.ac.at/monika.doerfler/NSPV.htm> under the name "Georgia On My Mind - Original". We consider only sample 33001 – 295144.

Since music signals are continuous signals of finite energy, it make sense to consider them in the framework of decomposition spaces. Moreover, the decomposition space norm constitutes a natural measure for such nonstationary signals, capable of detecting local signal changes as opposed to the standard L^p -norm.

In the following two sections we separately analyse the performance of a classical Gabor expansion and an adaptive nonstationary Gabor expansion. To analyse the performance of an expansion we use the relative root mean square (RMS) reconstruction error

$$\text{RMS}(f, f_{\text{rec}}) := \frac{\|f - f_{\text{rec}}\|_2}{\|f\|_2}.$$

As a general rule of thumb, an RMS error below 1% is hardly noticeable to the average listener. We measure the redundancy of a transform by

$$\frac{\text{number of coefficients}}{\text{length of signal}}.$$

For both transforms used below the redundancy is approximately 5/6 (only the coefficients corresponding to the positive frequencies are calculated for real valued signals).

For the implementation we use MATLAB 2017A and in particular we use the following two toolboxes: The LTFAT [33] (version 2.1.2 or above) available from <http://ltfat.github.io/> and the NSGToolbox [1] (version 0.1.0 or above) available from <http://nsg.sourceforge.net/>.

5.1 Classical Gabor frames

We construct the Gabor expansion using 2560 channels and a hop size of 1536. The window function is chosen as a Hanning window of length 2560 such that the resulting system constitutes a painless Gabor frame. The Gabor transform has a redundancy of ≈ 0.855 and the total number of Gabor coefficients is 224175 (of which 220332 are non-zero). Performing hard threshold, and keeping only the 13683 largest coefficients, we obtain an expansion with an RMS reconstruction error just below 1%. Spectrograms based on the original expansion and the thresholded expansion can be found in Figure 2.

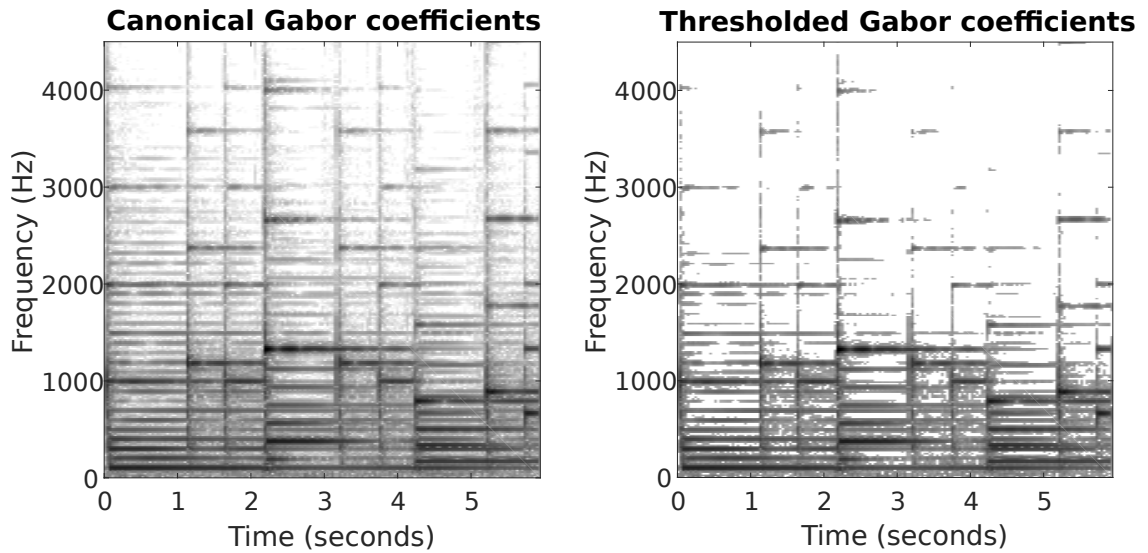


Fig. 2 Spectrograms based on full and thresholded Gabor expansion (RMS error just below 1%).

We first comment on the spectrogram based on the full Gabor expansion. The 9 "vertical stripes" correspond to the 9 onsets of the music (see Figure 1) and the "horizontal stripes" corresponds to the frequencies of the harmonics. We would like to remove the non-zero coefficients between the horizontal stripes as these are mainly caused by noise, blurry resolution, and the interaction between harmonics. In the thresholded spectrogram we have removed all coefficients with magnitudes smaller than a certain threshold. Keeping only around 6.1% of the original non-zero coefficients, we obtain an expansion which is capable of reproducing the original signal with an RMS reconstruction error smaller than 1%.

Based on the obtained results from Section 4 (in particular (17)) we expect the RMS error $E(N)$ to decrease as $N^{-\alpha}$, for some $\alpha \in \mathbb{R}^+$, with N being the number of non-zero coefficients. For different values of N , we have calculated $E(N)$ and used standard power regression to determine α . In Figure 3 we have collected selected values of $E(N)$ and N in a table and made a plot of the estimated power function.

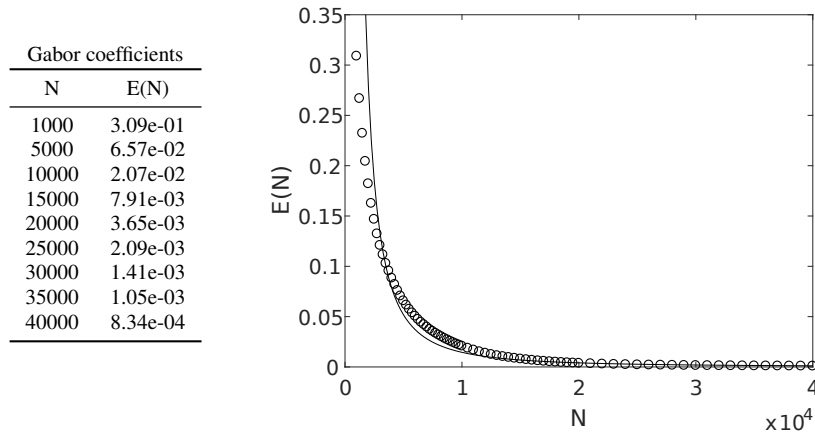


Fig. 3 RMS error $E(N)$ as a function of the number of non-zero coefficients N . Certain values are shown in the table to the left and power regression has been performed on considerably more values to the right.

The estimated value for α is 1.83616, which is a considerable faster rate than the one given in (17) (which belongs to $(0, 1/2]$). This illustrates that (17) only provides us with an *upper bound* on the approximation error - the actual error might be much smaller. We note that power regression seems to be a good fit for the data points in Figure 3. In the next section we perform similar experiments, replacing the Gabor frame with an adaptive NSGF.

5.2 Nonstationary Gabor frames

In order to obtain adaptive time-frequency representations with NSGF's, an adaptation procedure must be chosen. In this section we adopt the procedure from [1] and construct so-called *scale frames*. The idea is to calculate the onsets of the music piece, using a separate algorithm [8], and then use short window functions around the onsets and long window functions between the onsets. The space between two onsets is spanned in such a way that the window length first increases and then decreases.

The window functions are constructed as scaled versions of a fixed window prototype (a Hanning window was chosen for the implementation) and the scaling is done in a smooth way to avoid unnecessary jump in window size. Denoting the scale sequence by $\{s_n\} \subset \mathbb{N}$, we assume that $|s_n - s_{n-1}| \in \{0, 1\}$ for all n . The window functions are then given by

$$g_n(t) = \sqrt{2^{-s_n}} g(2^{-s_n} t),$$

with g denoting the window prototype. Assuming g has support in an interval of length 1, then g_n has support in an interval of length 2^{5n} . The construction is such that adjacent windows are either of the same length or one is twice as long as the other. The corresponding numbers of frequency channels are chosen such that the resulting system constitutes a painless NSGF. By construction, scale frames use very little overlap between the window functions to obtain a low redundancy.

For the actual implementation, we use 8 different Hanning windows with lengths varying from 192 (around the onsets) to $192 \cdot 2^7 = 24576$. For the piano music given in Figure 1, the nonstationary Gabor transform has a redundancy of ≈ 0.817 , which is comparable to that of the Gabor transform used in Section 5.1. The total number of coefficients is 214138 of which all are non-zero. Keeping only the 10371 largest coefficients we obtain an expansion with an RMS reconstruction error just below 1%. This is considerably fewer coefficients than needed for the standard Gabor expansion, which shows a natural sparseness of scale frames for this particular signal class. This property was already noted by the authors in [1]. Spectrograms based on the full expansion and the thresholded expansion can be found in Figure 4.

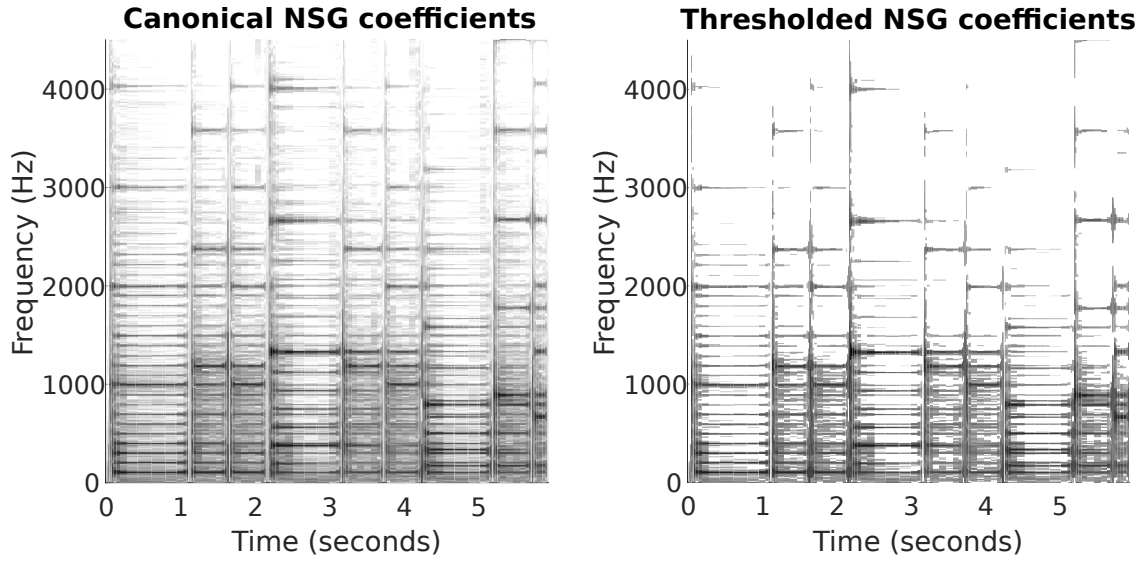


Fig. 4 Spectrograms based on full and thresholded nonst. Gabor expansion (RMS error just below 1%).

Just as for Figure 2, we note the 9 vertical stripes corresponding to the onsets and the horizontal stripes corresponding to the harmonics. However, in contrast to Figure 2, the horizontal stripes are no longer "rectangular". This reflects the adaptive behaviour of the window functions, resulting in a good time resolution around the 9 onsets and a good frequency resolution for the harmonics between the onsets. Considering the thresholded spectrogram to the right, we note that the approximation is particularly convincing for the three quarter notes (the longer ones), where the harmonics are easily accessible. Keeping only around 4.1% of the coefficients we obtain an expansion with an RMS reconstruction error smaller than 1%. Calculating the RMS error $E(N)$ for different values of N , the number of non-zero coefficients, and performing power regression, we obtain the plot shown in Figure 5.

The estimated value for rate of the approximation is $\alpha = 1.73346$, which is again much higher than the bound given in (17). The rate is a bit slower than for the classical Gabor frame, but the initial advance of the scale frame implies that all values in the table in Figure 5 are lower than those in the table in Figure 3. The power regression model also seems to be a good fit for the nonstationary case, aligning well with the set of data points.

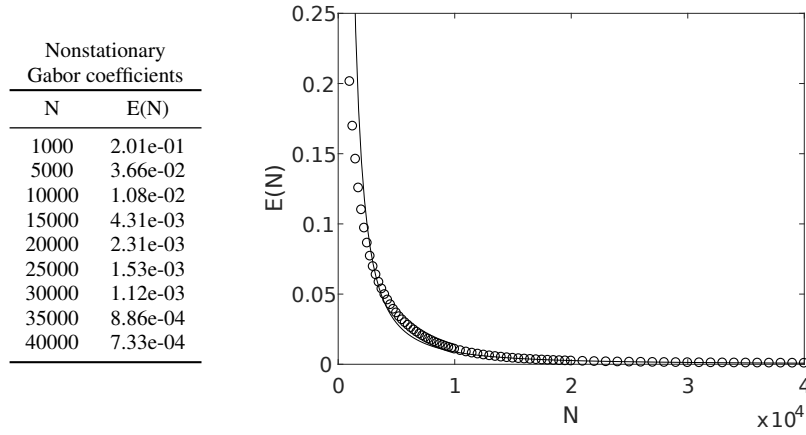


Fig. 5 RMS error $E(N)$ as a function of the number of non-zero coefficients N . Certain values are shown in the table to the left and power regression has been performed on considerably more values to the right.

6 Conclusion

We have provided a self-contained description of decomposition spaces on the time side, proving several important properties of such spaces. Given a painless nonstationary Gabor frame, with resolution varying in time, we have shown how to construct an associated decomposition space which characterizes the signals with sparse frame expansions. Based on this characterization we have proved an upper bound on the approximation error occurring when thresholding the coefficients of the frame expansion. The theoretical results have been complemented with numerical experiments, showing that the approximation error is indeed smaller than the theoretical upper bound. Using terminology from nonlinear approximation theory, we have proven a Jackson inequality for nonlinear approximation with certain nonstationary Gabor frames. It could be interesting to consider the inverse estimate, a so-called Bernstein inequality, providing us with a lower bound on the approximation error. The numerical experiments indeed suggests that the approximation error acts as a power function of the number of non-zero coefficients. Unfortunately, obtaining a Bernstein inequality for a redundant dictionary is in general beyond the reach of current methods [21].

A Proof of Theorem 1

Proof We will use the well known fact that

$$\int_{\mathbb{R}^d} u(x)^{-m} dx = \int_{\mathbb{R}^d} (1 + \|x\|_2)^{-m} dx < \infty, \quad m > d. \quad (18)$$

We prove each of the four statements separately and we write $D_{p,q}^s := D(\mathcal{Q}, L^p, \ell_{\omega^s}^q)$ to simplify notation.

1. Repeating the arguments from [3, Proposition 5.7], using Definition 2(2), we can show that

$$D_{p,\infty}^{s+\varepsilon} \hookrightarrow D_{p,q}^s \hookrightarrow D_{p,\infty}^s, \quad \varepsilon > d/q, \quad (19)$$

for any $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Hence, to prove Theorem 1(1) it suffices to show that $\mathcal{S}(\mathbb{R}^d) \hookrightarrow D_{p,\infty}^s \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ for any $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We first show that $\mathcal{S}(\mathbb{R}^d) \hookrightarrow D_{p,\infty}^s$. Since $\{\omega_T\}_{T \in \mathcal{T}}$ is \mathcal{Q} -moderate, and ψ_T is uniformly bounded, this result follows from (18) since

$$\begin{aligned} \omega_T^s \|\psi_T f\|_{L^p} &\leq C_1 \|u^s \psi_T f\|_{L^p} \leq C_2 \|u^s f\|_{L^p} \leq C_3 \|u^{s+r} f\|_{L^\infty} \\ &\leq C_3 \max_{|\beta| \leq N} \sup_{x \in \mathbb{R}^d} |u(x)^N \partial_x^\beta f(x)|, \quad f \in \mathcal{S}(\mathbb{R}^d), \end{aligned}$$

for $r > d/p$ and $N \geq s + r$. To show that $D_{p,\infty}^s \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ we define $\widetilde{\psi}_T := \sum_{T' \in \widetilde{T}} \psi_{T'}$. Given $f \in D_{p,\infty}^s$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, Hölder's inequality yields

$$\begin{aligned} |\langle f, \varphi \rangle| &= \left| \sum_{T \in \mathcal{T}} \langle \psi_T f, \widetilde{\psi}_T \varphi \rangle \right| \leq \sum_{T \in \mathcal{T}} \|\psi_T f\|_{L^p} \|\widetilde{\psi}_T \varphi\|_{L^{p'}} \\ &\leq \sum_{T \in \mathcal{T}} \|\psi_T f\|_{L^p} \|\widetilde{\psi}_T \varphi\|_{L^{p'}} \leq \|f\|_{D_{p,\infty}^s} \sum_{T \in \mathcal{T}} \omega_T^{-s} \|\widetilde{\psi}_T \varphi\|_{L^{p'}}, \end{aligned} \quad (20)$$

with $1/p + 1/p' = 1$. Applying (2) we get

$$\begin{aligned} \sum_{T \in \mathcal{T}} \omega_T^{-s} \|\widetilde{\psi}_T \varphi\|_{L^{p'}} &\leq \left\| \left\{ \sum_{T' \in \widetilde{T}} \|\psi_{T'} \varphi\|_{L^{p'}} \right\}_{T \in \mathcal{T}} \right\|_{\ell^1_{\omega^{-s}}} = \left\| \left\{ (\|\psi_T \varphi\|_{L^{p'}})^+ \right\}_{T \in \mathcal{T}} \right\|_{\ell^1_{\omega^{-s}}} \\ &\leq C_+ \left\| \left\{ \|\psi_T \varphi\|_{L^{p'}} \right\}_{T \in \mathcal{T}} \right\|_{\ell^1_{\omega^{-s}}} = C_+ \|\varphi\|_{D_{p',1}^{-s}}. \end{aligned} \quad (21)$$

Now, (19) implies $\|\varphi\|_{D_{p',1}^{-s}} \leq C \|\varphi\|_{D_{p',\infty}^{s-s}}$ for $\varepsilon > d$. Hence, since we have already shown that $\mathcal{S}(\mathbb{R}^d) \hookrightarrow D_{p,\infty}^s$, we conclude from (20) and (21)

that $D_{p,\infty}^s \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. This proves Theorem 1(1).

2. Theorem 1(2) follows from Theorem 1(1) and the arguments in [3, Page 150].

3. To prove Theorem 1(3) we let $f \in D_{p,q}^s$ and choose a function $I \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq I(x) \leq 1$ and $I(x) \equiv 1$ on some neighbourhood of $x = 0$. Since $\text{supp}(I)$ is compact we can choose a finite subset $T^* \subset \mathcal{T}$ such that $\text{supp}(I) \subset \cup_{T \in T^*} Q_T$ and $\sum_{T \in T^*} \psi_T(x) \equiv 1$ on $\text{supp}(I)$. Hence, with $\widetilde{f} := If$ we get

$$\|\widetilde{f}\|_{L^p} = \left\| \sum_{T \in T^*} \psi_T If \right\|_{L^p} \leq \sum_{T \in T^*} \|\psi_T f\|_{L^p} < \infty, \quad (22)$$

since $f \in D_{p,q}^s$. Now, let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \varphi(x) \leq 1$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. Also, for $\varepsilon > 0$ define $\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(x/\varepsilon)$ and let $\widetilde{f}_\varepsilon := \varphi_\varepsilon * \widetilde{f} \in \mathcal{S}(\mathbb{R}^d)$. It follows from (22) and a standard result on L^p -spaces [28, Theorem 2.16 on page 64] that

$$\|\widetilde{f} - \widetilde{f}_\varepsilon\|_{D_{p,q}^s} \leq C \left\| \left\{ \|\widetilde{f} - \widetilde{f}_\varepsilon\|_{L^p} \right\}_{T \in \mathcal{T}} \right\|_{\ell^q_{\omega^s}} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Hence, the proof is done, if we can show that $\|f - \widetilde{f}\|_{D_{p,q}^s}$ can be made arbitrary small by choosing \widetilde{f} appropriately. To show this, we define $T_\circ := \{T \in \mathcal{T} \mid I(x) \equiv 1 \text{ on } \text{supp}(\psi_T)\}$. Denoting its complement by T_\circ^c we get

$$\|f - \widetilde{f}\|_{D_{p,q}^s} \leq 2 \left\| \left\{ \|\psi_T f\|_{L^p} \right\}_{T \in T_\circ^c} \right\|_{\ell^q_{\omega^s}}.$$

Finally, since $f \in D_{p,q}^s$, we can choose $\text{supp}(I)$ large enough, such that $\|f - \widetilde{f}\|_{D_{p,q}^s} < \varepsilon$ for any given $\varepsilon > 0$. This proves Theorem 1(3).

4. To prove Theorem 1(4) we first note that $(D_{p,q}^s)' \subset \mathcal{S}'(\mathbb{R}^d)$ since $\mathcal{S}(\mathbb{R}^d) \subset D_{p,q}^s$. Furthermore, by Remark 2 we may assume the same BAPU $\{\psi_T\}_{T \in \mathcal{T}}$ is used for both $D_{p,q}^s$ and $D_{p',q'}^{-s}$. Let us first show that $D_{p',q'}^{-s} \subseteq (D_{p,q}^s)'$. Given $\sigma \in D_{p',q'}^{-s}$ and $f \in D_{p,q}^s$, applying (2) and Hölder's inequality twice yield

$$\begin{aligned} |\langle f, \sigma \rangle| &= \left| \sum_{T \in \mathcal{T}} \langle \widetilde{\psi}_T f, \psi_T \sigma \rangle \right| \leq \sum_{T \in \mathcal{T}} \|\widetilde{\psi}_T f\|_{L^p} \|\psi_T \sigma\|_{L^{p'}} \\ &\leq \sum_{T \in \mathcal{T}} \left(\omega_T^s \sum_{T' \in \widetilde{T}} \|\psi_{T'} f\|_{L^p} \right) (\omega_T^{-s} \|\psi_T \sigma\|_{L^{p'}}) \\ &\leq \left\| \left\{ (\|\psi_T f\|_{L^p})^+ \right\}_{T \in \mathcal{T}} \right\|_{\ell^q_{\omega^s}} \left\| \left\{ \|\psi_T \sigma\|_{L^{p'}} \right\}_{T \in \mathcal{T}} \right\|_{\ell^{q'}_{\omega^{-s}}} \leq C_+ \|f\|_{D_{p,q}^s} \|\sigma\|_{D_{p',q'}^{-s}}. \end{aligned} \quad (23)$$

To prove that $(D_{p,q}^s)' \subseteq D_{p',q'}^{-s}$ we define the space

$$\ell^q(L^p) := \left\{ \{f_T\}_{T \in \mathcal{T}} \subset \mathcal{S}'(\mathbb{R}^d) \mid \|\{f_T\}_{T \in \mathcal{T}}\|_{\ell^q(L^p)} := \left\| \left\{ \|f_T\|_{L^p} \right\}_{T \in \mathcal{T}} \right\|_{\ell^q} < \infty \right\}. \quad (24)$$

With this notation we get

$$\|f\|_{D_{p,q}^s} = \left\| \left\{ \omega_T^s \|\psi_T f\|_{L^p} \right\}_{T \in \mathcal{T}} \right\|_{\ell^q} = \left\| \left\{ \omega_T^s \psi_T f \right\}_{T \in \mathcal{T}} \right\|_{\ell^q(L^p)},$$

for all $f \in D_{p,q}^s$. Since $f \rightarrow \{\omega_T^s \psi_T f\}_{T \in \mathcal{T}}$ defines an injective mapping from $D_{p,q}^s$ onto a subspace of $\ell^q(L^p)$, every $\sigma \in (D_{p,q}^s)'$ can be interpreted as a functional on that subspace. By the Hahn-Banach theorem, σ can be extended to a continuous linear functional on $\ell^q(L^p)$ where the norm of σ is preserved. It thus follows from [36, Proposition 2.11.1 on page 177] that for $f \in D_{p,q}^s$ we may write

$$\sigma(f) = \int_{\mathbb{R}^d} \sum_{T \in \mathcal{T}} \sigma_T(x) \omega_T^s \psi_T(x) f(x) dx, \quad \text{where} \quad (25)$$

$$\{\sigma_T(x)\}_{T \in \mathcal{T}} \in \ell^{q'}(L^{p'}), \quad \text{and} \quad \|\sigma\|_* = \|\{\sigma_T\}_{T \in \mathcal{T}}\|_{\ell^{q'}(L^{p'})}, \quad (26)$$

with $\|\sigma\|_* := \sup_{\|f_T\|_{\ell^q(L^p)}=1} |\sigma(\{f_T\})|$ denoting the standard norm on $(\ell^q(L^p))'$. From (25) we conclude that the proof is done if we can show that $\sum_{T \in \mathcal{T}} \sigma_T(x) \omega_T^s \psi_T(x) \in D_{p',q'}^{-s}$. This follows from (2) since

$$\begin{aligned} \left\| \sum_{T \in \mathcal{T}} \sigma_T \omega_T^s \psi_T \right\|_{D_{p',q'}^{-s}} &= \left\| \left\{ \left\| \psi_T \left(\sum_{T' \in \mathcal{T}} \sigma_{T'} \omega_{T'}^s \psi_{T'} \right) \right\|_{L^{p'}} \right\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^{-s}}^{q'}} \\ &\leq C_1 \left\| \left\{ \|\sigma_T \omega_T^s \psi_T\|_{L^{p'}} \right\}_{T \in \mathcal{T}} \right\|_{\ell_{\omega^{-s}}^{q'}} \leq C_2 \left\| \left\{ \|\sigma_T\|_{L^{p'}} \right\}_{T \in \mathcal{T}} \right\|_{\ell^{q'}} \\ &= C_2 \|\{\sigma_T\}_{T \in \mathcal{T}}\|_{\ell^{q'}(L^{p'})} = \|\sigma\|_*, \end{aligned}$$

where we use (26) in the last equation. This proves Theorem 1(4). \square

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